23-4-304 Yuigahama 2-chome, Kamakura-shi 248-0014, Japan

A set of orthonormal trigonometric function series and its applications

Yoshimutsu Hirata

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Abstract

This paper presents the orthonormal function that is experessed by the finite series of trigonometric functions. The function provides the mathematical tool of spectrum dispersion and assembling, which is available for bandwidth compression. Some applications are also described in the paper.

Key words: orthonormal function, orthogonal function, orthogonal function series, spectrum dispersion, spectrum approximation
Mathematics Subject Classification (2010): 42C10

1 Introduction

The orthogonal system of trigonometric functions is widely used for the analysis of time series. In signal processing, especially in speech signal analysis and synthesis, the fast Fourier transform (FFT) is the most important mathematical tool [1], [2]. This paper presents another orthogonal system of functions which are expressed by the finite series of trigonometric functions, The quasi-spectrum that is given by the transform of time series using the proposed orthogonal functions forms a contrast to the FFT spectrum, which provides a new tool for signal processing.

2 Orthonormal function series

Let $E(n,x)$ be a complex function of the real argument x $(0 \le x < 1)$, which is expressed by

$$E(n,x) = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} r_{nm} \mathrm{e}^{j2\pi mx}, \tag{1}$$

where $r_{nm} = 1$ or -1 $(m, n = 1, 2, \cdots, N)$, N is the power of 2, $j = \sqrt{-1}$, and

$$\sum_{m=1}^{N} r_{nm} r_{km} = \begin{cases} 0, & (k \neq n) \\ N, & (k = n) \end{cases} \tag{2}$$

for all n. Hence, the inner product

$$(E_n, E_k) = \int_0^1 E(n, x) E(k, -x) dx = 0, \tag{3}$$

for $k \neq n$, and

$$(E_n, E_n) = 1, \tag{4}$$

so that we have a set of orthonormal functions: $E(1, x), E(2, x), \cdots, E(N, x)$. These functions are expressed by the finite series of N elements,

$$\mathrm{e}^{j2\pi x}, \mathrm{e}^{j4\pi x}, \cdots, \mathrm{e}^{j2\pi N x}.$$

The sequence $\{r_{nm}\}$, or $\{r_{mn}\}$, that satisfies Eq, (2) is given by the discrete Walsh function of sample length N [3], [4]. Thus, from N linear equations given by eq. (1), we get

$$\mathrm{e}^{j2\pi m x} = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} r_{mn} E(n, x). \tag{5}$$

As a corollary, we can express

$$E(n, x) = C(n, x) + jS(n, x) \tag{6}$$

where,

$$C(n, x) = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} r_{nm} \cos 2\pi m x, \tag{7}$$

$$S(n, x) = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} r_{nm} \sin 2\pi m x. \tag{8}$$

We see that

$$\begin{aligned} (C_n, C_k) &= 0, \ (S_n, S_k) = 0, \text{ for } k \neq n \\ (C_n, S_k) &= 0, \text{ for all } k, n \\ (C_n, C_n) &= 1/2, \ (S_n, S_n) = 1/2. \end{aligned} \tag{9}$$

As to eq,(5), we have by eqs. (6) - (8),

$$\cos 2\pi m x = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} r_{mn} C(n, x), \tag{10}$$

$$\sin 2\pi m x = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} r_{mn} S(n, x). \tag{11}$$

3 Applications

Let $f(x)$ be a function of x $(0 \leq x < 1)$ which is expressed by a Fourier series of the form

$$f(x) = a_0 + \sum_{m=1}^{N} a_m \cos 2\pi m x + \sum_{m=1}^{N} b_m \sin 2\pi m x \tag{12}$$

where the coefficients a_m and b_m are given by the well known formulas. The function $f(x)$ is expressed by

$$f(x) = a_0 + \sum_{n=1}^{N} u_n C(n,x) + \sum_{n=1}^{N} v_n S(n,x), \tag{13}$$

where

$$u_n = 2\int_0^1 f(x)C(n,x)dx, \tag{14}$$

$$v_n = 2\int_0^1 f(x)S(n,x)dx. \tag{15}$$

Thus, by eqs. (7) and (8), we have

$$u_n = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} r_{nm} a_m, \tag{16}$$

$$v_n = \frac{1}{\sqrt{N}} \sum_{m=1}^{N} r_{nm} b_m. \tag{17}$$

Hence

$$a_m = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} r_{mn} u_n. \tag{18}$$

$$b_m = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} r_{mn} v_n. \tag{19}$$

Let $F(m)$ be the power spectral density, or simply the spectrum, of $f(x)$ which is expressed by

$$F(m) = a_m^2/2 + b_m^2/2, \tag{20}$$

and let $Y(n)$ be the quasi-spectrum which is defined by

$$Y(n) = u_n^2/2 + v_n^2/2. \tag{21}$$

Thus, if we put $F(p) \neq 0$ and $F(k) = 0$ for $k \neq p$, we have, by eqs. (16) and (17),

$$Y(n) = F(p)/N, \tag{22}$$

for all n, and if we put $F(p) \neq 0$, $F(q) \neq 0$ and $F(k) = 0$ for $k \neq p, q$, we have

$$Y(n) = F(p)/N + F(q)/N + r_{np}r_{nq}(a_p a_q + b_p b_q), \tag{23}$$

and so on. Fig. 1 shows some examples of $Y(n)$ corresponding to $F(m)$. We see that the quasi-spectrum forms contrast to the Fourier spectrum. By eqs. (12) and (13), we have

$$\sum_{m=1}^{N} F(m) = \sum_{n=1}^{N} Y(n). \tag{24}$$

Thus, if

$$Y(p) \gg Y(n), (n \neq p) \tag{25}$$

we get

$$f(x) \cong a_0 + u_p C(p, x) + v_p S(p, x). \tag{26}$$

Furthermore, if

$$Y(p), Y(q) \gg Y(k), (k \neq p, q) \tag{27}$$

we get

$$f(x) \cong a_0 + u_p C(p, x) + v_p S(p, x) + u_q C(q, x) + v_q S(q, x) = g(x), \tag{28}$$

where $g(x)$ is the aporoximate function of $f(x)$. Thus, by Parseval's equation,

$$\int_0^1 \{f(x) - g(x)\}^2 \, dx = \sum_{m=1}^{N} R(m), \tag{29}$$

where $R(m)$ is the spectrum of the error, $f(x) - g(x)$, so that

$$\sum_{m=1}^{N} R(m) = \sum_{m=1}^{N} F(m) - Y(p) - Y(q). \tag{30}$$

The process of the approximation described above is illustrated in Fig.2 where (a) shows the spectrum $F(m)$ to be approximated and the coefficient $a_m (b_m = 0)$, (b) the quasi-spectrum $Y(n)$ corresponding to $F(m)$ and the coefficient $u_n (v_n = 0)$, (c) the approximate spectrum $G(m)$ and the coefficient a_m, which are given by eq. (18) where u_7=0.47, u_5=0.11, r_{m7}=(1, -1, 1, -1, -1, 1, -1, 1) and r_{m5}=(1, -1, -1, 1, 1, -1, -1, 1), and (d) the error spectrum $R(m)$ and the coefficient a_m.

The author can not afford to present practical examples which would make it easier for the reader to understand the properties of the proposed orthogonal functions and their utility for representing a function. We see, however, that the approximate spectrum, such as shown in Fig.2(c), generally covers the whole frequency range of an original spectrum, which offers a new tool for signal processing.

4 Conclusion

The orthogonal function proposed in this paper provides the mathematical tool of spectrum dispersion and assembling, which is available for the bandwidth compression of time series where the clutter spectrum needs be approximated [5],[6]. The coefficients (u_n,v_n) are derived from the FFT coefficients (a_m,b_m) and vice versa, which makes it feasible to apply the proposed functions to practical use.

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6 Appendix

We see that

$$\mathrm{e}^{j2\pi Nx}E(n,x) = \frac{1}{\sqrt{N}}\sum_{m=1}^{N} r_{nm}\mathrm{e}^{j2\pi(m+N)x}. \tag{31}$$

Thus, if we put

$$E(n+N,x) = \mathrm{e}^{j2\pi Nx}E(n,x), \tag{32}$$

we have

$$\mathrm{e}^{j2\pi(m+N)x} = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} r_{mn}E(n+N,x). \tag{33}$$

This relationship is available when we deal with $2N$ coefficients by eq. (16), for example, i.e., $(u_1, u_2, \cdots u_N)$ are given by ($a_1, a_2, \cdots a_N$) and $(u_{n+1}, u_{n+2}, \cdots, u_{2N})$ by $(a_{n+1}, a_{n+2}, \cdots, a_{2N})$.

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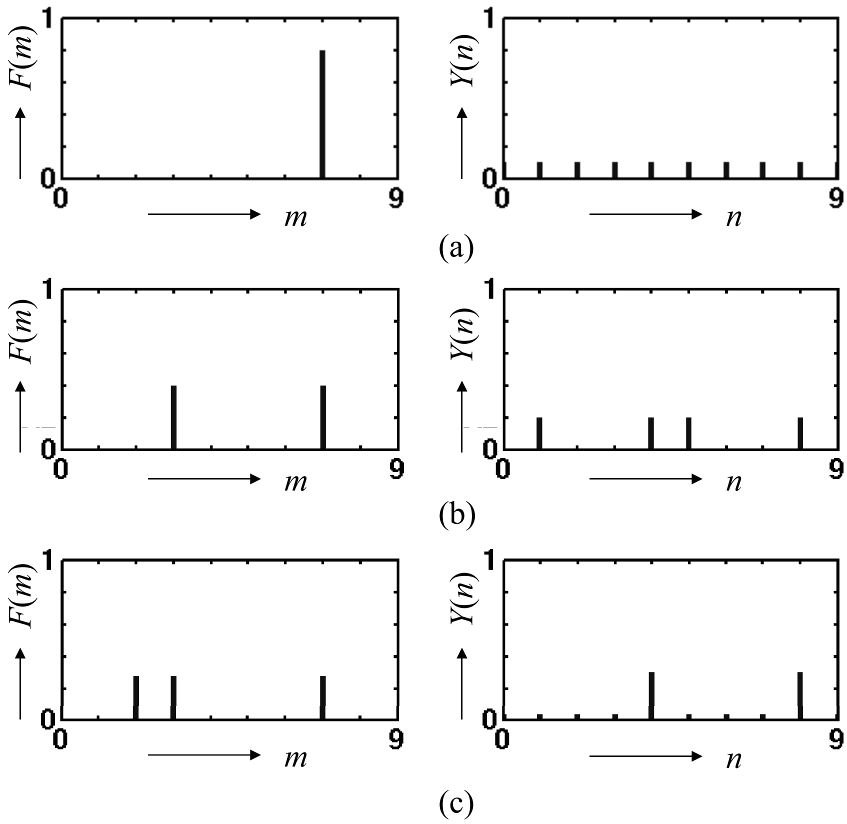

Figure 1: Spectrum $F(m)$ and quasi-spectrum $Y(n)$; (a) $F(7)$=0.8, (b) $F(3)$=$F(7)$=0.4 and (c) $F(2)$=$F(3)$=$F(7)$=0.27, (N=8).

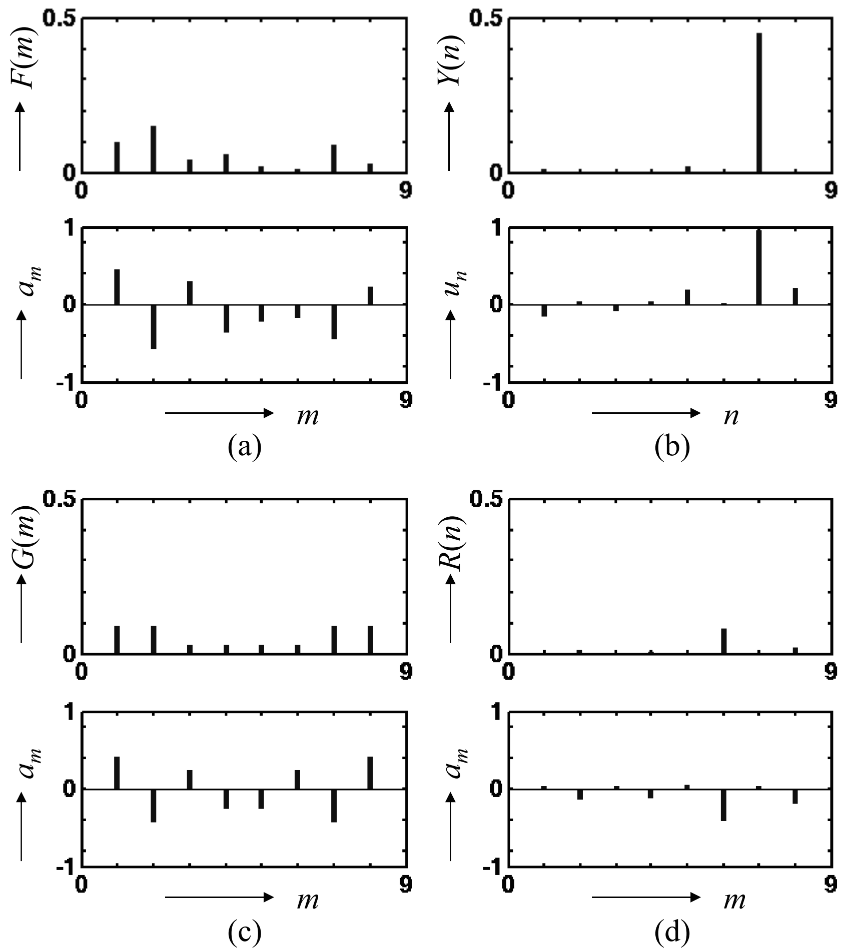

Figure 2: Spectrum $F(m)$ and the coefficient a_m, (b) the quasi-spectrum Y(n) and the coefficient u_n, (c) the approximate spectrum $G(m)$ and the coefficient a_m, and (d) the error spectrum $R(m)$ and the coefficient a_m, ($b_m = 0$, $N = 8$).