Bell inequality violations with nanomechanical resonators

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- Review of Bell inequalities
- Device and model
- Bell inequality for binned quadrature measurements
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- Conditions for obtaining a violation of the Bell inequality with the mechanical system under consideration
- Conclusions
Some recent progress on nanomechanical systems in the quantum regime

UCSB
O’Connell et al., Nature (2010)

NIST
Teufel et al., Nature (2011)

NIST
Palomaki et al., Science (2013)

NIST
Palo et al., Nature (2013)

Caltech
Chan et al., Nature (2011)
Objectives

Assuming the resonator can be cooled down its quantum ground state, we are interested in:

- generating and detecting nonclassical states

Several possible methods:

- Coupling to a qubit, or other nonlinear system, coupling to auxiliary optical cavities, applying nonlinear potential, using intrinsic nonlineararities, nonlinear dissipation/engineered environments, ...

Can produce different types of nonclassical states:

- Subpoissionian, quadrature squeezed, superposition states, two-mode squeezed states, entangled states, etc.

Can be demonstrated and quantified with:

- Negative Wigner functions, variances below the vacuum level, entanglement witnesses and measures, such as negativity, entanglement entropy, Bell inequalities.
Review of the CHSH Bell inequality

CHSH = Clauser-Horne-Shimony-Holt
PRL 23, 880 (1969)

\[ |\psi\rangle \propto |0, 0\rangle + |1, 1\rangle \]

For local hidden variable theories \( \Rightarrow |B_{\text{CHSH}}| \leq 2 \)

\[ B_{\text{CHSH}} = E(\theta, \phi) - E(\theta', \phi) + E(\theta, \phi') + E(\theta', \phi') \]

\[ E(\theta, \phi) = P_{11}(\theta, \phi) + P_{00}(\theta, \phi) - P_{10}(\theta, \phi) - P_{01}(\theta, \phi) \]

\( P_{\alpha\beta}(\theta, \phi) = \) probability to measure \( \alpha \) and \( \beta \) at \( D_1 \) and \( D_2 \)
Conceptual device

See e.g., I. Mahboob et al, PRL 110, 127202 (2013):
$H = \sum_k \omega_k a_k^\dagger a_k + \sum_{klm} \beta_{klm} x_k x_l x_m + O(x_k x_l x_m x_n)$

- Second order coupling terms eliminated by choice of basis
- Third order nonlinearity can occur in asymmetric resonators
- Higher order nonlinearities can also occur, for example in resonators under strain (fourth terms)

Choice of modes

Among the very large number of modes in the resonator, we choose to work with three modes that:

- Satisfy the mode matching condition: \( \omega_1 + \omega_2 \approx \omega_0 \)
- With decay rates: \( \gamma_1, \gamma_2 \ll \gamma_0 \)
- As large three-mode coupling as possible: \( \beta_{012} > \gamma_1, \gamma_2 \)
Parametric oscillator regime

Under the rotating wave approximation, the three selected modes are described by the Hamiltonian:

\[
H = \sum_{k=0}^{2} \omega_k a_k^\dagger a_k + i \kappa (a_1^\dagger a_2^\dagger a_0 - a_1 a_2 a_0^\dagger)
\]

\(a_1 = \text{signal mode}\)
\(a_2 = \text{idler mode}\)
\(a_0 = \text{pump mode}\)
Parametric oscillator regime

We drive the pump mode with frequency $\omega_L$ and move to the rotating frame in which the driving term is time-independent:

$$H = \Delta_L a_0^\dagger a_0 + \sum_{k=1,2} \Delta_k a_k^\dagger a_k + i\kappa(a_1^\dagger a_2^\dagger a_0 - a_1 a_2 a_0^\dagger)$$

$$-i(E a_0^\dagger - E^* a_0)$$

$$\Delta_L = \omega_0 - \omega_L, \Delta_1 = \Delta_2 = (\Delta_0 - \Delta_L)/2$$
We model the dynamics of the three coupled modes with a standard Lindblad master equation, where each mode has single-photon relaxation and thermal excitation:

\[
\dot{\rho} = -i[H, \rho] + \sum_{k=0,1,2} \gamma_k \left\{ (N_k + 1) \mathcal{D}[a_k] + N_k \mathcal{D}[a_k^\dagger] \right\} \rho
\]

\[
\mathcal{D}[a_k] \rho = a_k \rho a_k^\dagger - \frac{1}{2} a_k^\dagger a_k \rho - \frac{1}{2} \rho a_k a_k^\dagger
\]

\(\gamma_k\) = dissipation rate for mode \(k\)

\(N_k\) = average thermal photon occupation number of mode \(k\)
Adiabatic elimination of the pump mode

Assuming that the dissipation rate of the pump mode dominates over the coherent dynamics

\[ \gamma_0 \gg \langle H \rangle \]

we can adiabatically eliminate the pump mode and obtain an effective two-mode model, which includes a correlated two-photon dissipation term:

\[
\dot{\rho} = -i[H', \rho] + \gamma \mathcal{D}[a_1 a_2] \rho \\
+ \sum_{k=1,2} \gamma_k \left\{ (N_k + 1) \mathcal{D}[a_k] + N_k \mathcal{D}[a_k^\dagger] \right\} \rho
\]
Effective two-mode model

Hamiltonian and master equation for the effective two-mode model:

\[ H' = \sum_{k=1,2} \Delta_k a_k^\dagger a_k + i(\mu a_1^\dagger a_2^\dagger - \mu^* a_1 a_2) + \chi a_1^\dagger a_1 a_2^\dagger a_2 \]

\[ \dot{\rho} = -i[H', \rho] + \gamma \mathcal{D}[a_1 a_2] \rho + \sum_{k=1,2} \gamma_k \left\{ (N_k + 1) \mathcal{D}[a_k] + N_k \mathcal{D}[a_k^\dagger] \right\} \rho \]

\[ \mu = \frac{E \kappa}{\gamma_0/2 + i \Delta_L} \quad \chi = -\frac{\kappa^2 \Delta_L}{|\gamma_0/2 + i \Delta_L|^2} \quad \gamma = \frac{\kappa^2 \gamma_0/2}{|\gamma_0/2 + i \Delta_L|^2} \]
Steady state for ideal case

We cannot solve the dynamics or the steady state analytically for the full model, but for the case

\[ \gamma_1 = \gamma_2 = 0 \]

we have the steady state

\[
\rho = \frac{1}{I_0(2r^2)} \sum_{m,n} \frac{r^{2m+2n}}{m!n!} |m, m\rangle \langle n, n| 
\]

\( I_0 = \) zeroth order modified Bessel function

\( r = \sqrt{2E/\kappa} \)
The steady state for this effective two-mode model is:

- Symmetric with respect to the two modes
- Each mode individually has a positive Wigner function
- But there are strong inter-mode quadrature correlations

We can therefore expect that the two modes are entangled, and may violate a continuous-variable Bell inequality for quadrature measurements for the two modes.

\[ \kappa = 0.15, \ E = 0.094, \ \gamma_0 = 1.0, \ \gamma_1 = \gamma_2 = 0, \ \text{and} \ \Delta_0 = \Delta_L = 0 \]
Transient dynamics (from ground state)

When starting in the ground state, turning on the pump field drives the system such that

- the photon number increase
- single-mode quadrature variances increase
- one of the quadrature difference variances decrease below the vacuum fluctuation level
- Entanglement is formed between the two modes (non-zero logarithmic negativity)
Bell inequality for quadrature measurements

- The Bell inequality was originally developed for bipartite measurements, but in harmonic systems we have continuous and unbound measurement outcomes.
- However, we can turn a continuous measurement outcome to a discrete one by binning the results:

\[ x_1 > 0 \]

\[ x_1 < 0 \]

Gilchrist et al, PRL 1998, Munro PRA 1999. See also Wegner et al., PRA 2003
Quadrature binning I

\[ P_{\alpha\beta}(\theta, \phi) = \int_{L(\alpha)}^{U(\alpha)} \int_{L(\beta)}^{U(\beta)} d^2X \rho(X_1^\theta, X_2^\phi)[\rho] \]

\[ L(\alpha) = \begin{cases} 0 & \text{if } \alpha = 1 \\ -\infty & \text{if } \alpha = 0 \end{cases} \]

\[ U(\alpha) = \begin{cases} \infty & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha = 0 \end{cases} . \]

Gilchrist et al, PRL 1998, Munro PRA 1999. See also Wegner et al., PRA 2003
Quadrature binning II

\[ P_{\alpha\beta}(\theta, \phi) = \int_{L(\alpha)} U(\alpha) \int_{L(\beta)} U(\beta) \ d^2 X p(X_1^\theta, X_2^\phi)[\rho] \]

\[ p(X_1^\theta, X_2^\phi)[\rho] = \left\langle X_1^\theta, X_2^\phi \middle| \rho \middle| X_1^\theta, X_2^\phi \right\rangle \]

\[ = \sum_{m,n,p,q} \rho(m,n),(p,q) \frac{e^{-i(m\theta+n\phi)} e^{i(p\theta+q\phi)}}{\pi \sqrt{2m+n+p+q} m! n! p! q!} \times \]

\[ e^{-x_1^2} e^{-x_2^2} H_m(X_1) H_n(X_2) H_p(X_1) H_q(X_2) \]

\[ \rho = \sum_{m,n,p,q} \rho(m,n),(p,q) \left| m, n \right\rangle \left\langle p, q \right| \]

Gilchrist et al, PRL 1998, Munro PRA 1999. See also Wegner et al., PRA 2003
Bell inequality for quadrature measurements

- The CHSH Bell inequality for the case of binned quadrature measurements:

\[ |B_{\text{CHSH}}| \leq 2 \]

\[ B_{\text{CHSH}} = E(\theta, \phi) - E(\theta', \phi) + E(\theta, \phi') + E(\theta', \phi') \]

\[ E(\theta, \phi) = P_{11}(\theta, \phi) + P_{00}(\theta, \phi) - P_{10}(\theta, \phi) - P_{01}(\theta, \phi) \]

\[ P_{\alpha\beta}(\theta, \phi) = \int_{L(\alpha)}^{U(\alpha)} \int_{L(\beta)}^{U(\beta)} d^2X p(X_1^\theta, X_2^\phi)[\rho] \]

Gilchrist et al, PRL 1998, Munro PRA 1999. See also Wegner et al., PRA 2003
Optimal Bell violations with quadrature binning with the NEMS model

In the ideal case without single photon dissipation processes

\[ \gamma_1 = \gamma_2 = 0 \]

we obtain, in the steady state, the following equations for the optimal Bell violation

\[ I_0(2r^2) \frac{dG(r)}{dr} = 4r^2 I_1(2r^2)G(r) \]

\[
G(r) = \sum_n \sum_{m>n} \frac{8(2r^2)^{n+m\pi}}{(n!m!)^2(n-m)^2} \left[ \mathcal{F}(n, m) - \mathcal{F}(m, n) \right]^2 \times \left\{ 3 \cos \left[ (n-m)\chi \right] - \cos \left[ 3\chi(n-m) \right] \right\} \\
\mathcal{F}(n, m) = \left[ \Gamma \left( \frac{1}{2} - \frac{n}{2} \right) \Gamma \left( -\frac{m}{2} \right) \right]^{-1}
\]

Munro PRA 1999
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we obtain, in the steady state, the following equations for the optimal Bell violation

\[ I_0(2r^2) \frac{dG(r)}{dr} = 4r^2 I_1(2r^2) G(r) \]

This equations cannot be solved analytically, but numerical we obtain the solution:

\[ r_{\text{opt}} \approx 1.12 \]

From which we obtain the optimal driving strength to maximize the violation, given a fixed mode interaction strength:

\[ E_{\text{opt}} = r_{\text{opt}}^2 \kappa / 2 \]
In the presence of single photon dissipation:

\[ \gamma_1, \gamma_2 > 0 \]

There is no violation of the Bell inequality in the steady state, but there is a region of violation during the transient dynamics from the ground state to the non-violating steady state.
Regions of Bell inequality violations
Conditions for Bell inequality violation

In order to violate the Bell inequality with binned quadrature measurement and the model under consideration, we require:

- $E \approx E_{opt} = r_{opt}^2 \kappa / 2$
- $\gamma_1, \gamma_2 \ll \kappa^2 / \gamma_0$
- sufficiently fast and efficient quadrature measurements.
Temperature dependence

N = average number of thermal photons in when driving is applied

Very good ground state cooling is required for observing the Bell inequality violations in this nanomechanical system.
Conclusions

- We investigated generating nonclassical state and violating the Bell inequality for binned quadrature measurements with a nanomechanical resonator.
- We show that there is a regime in parameter space that should allow Bell inequality violations with a mechanical resonator.
- We identify ideal conditions for producing such violations.